Assignment 11

This homework is due *Tuesday* Dec 4.

There are total 32 points in this assignment. 29 points is considered 100%. If you go over 29 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 5.4, 6.1 in Bartle–Sherbert.

(1) [2pt] (5.4.2) Show that function $f(x) = 1/x^2$ is uniformly continuous on $A = [1, \infty)$, but that it is not uniformly continuous on $B = (0, \infty)$.

(2) (5.4.3) Use the Nonuniform Continuity Criterion to show that the following functions are not uniformly continuous on the given sets.
  (a) [2pt] $f(x) = x^2$, $A = [0, \infty)$.
  (b) [2pt] $g(x) = \sin(1/x)$, $B = (0, \infty)$.

(3) (a) [3pt] (5.4.6) Show that if f and g are uniformly continuous on $A \subseteq \mathbb{R}$, and if they are *both* bounded on A, then their product fg is uniformly continuous on A.
  (b) [3pt] (5.4.7) If $f(x) = x$ and $g(x) = \sin x$, show that both f and g are uniformly continuous on $\mathbb{R}$, but their product fg is not uniformly continuous on $\mathbb{R}$.
  Comment. This shows that only one function of f, g needs to be unbounded to fail assertion of 3a.

(4) (Part of 6.1.1) Use the definition to find derivative of each of the following functions:
  (a) [2pt] $f(x) = x^3$, $x \in \mathbb{R}$,
  (b) [2pt] $f(x) = 1/\sqrt{x}$, $x > 0$.
  (*Hint:* You can use any of the definitions of the derivative that were discussed in class, but the shortest here probably is the limit of ratio definition.)

(5) [2pt] ($\sim$6.1.2) Show that $f(x) = x^{1/10}$, $x \in \mathbb{R}$, is not differentiable at $x = 0$.

(6) [3pt] ($\sim$6.1.4) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$ for x rational, $f(x) = 0$ for x irrational. Show that f is differentiable at $x = 0$, and find $f'(0)$. (Hint: Use the limit definition of derivative.)

(7) [3pt] (6.1.7) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $c = 0$ and that $f(c) = 0$. Show that $g(x) = |f(x)|$ is differentiable at c if and only if $f'(c) = 0$.

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(8) [3pt] (6.1.10) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2 \sin(1/x^2)$ for $x \neq 0$, and $g(0) = 0$. Show that g is differentiable for all $x \in \mathbb{R}$. Also show that the derivative g' is not bounded on the interval $[-1, 1]$.

(9) [2pt] (6.1.14) Given that the function $h(x) = x^3 + 2x + 1$, $x \in \mathbb{R}$, has an inverse h^{-1} on $\mathbb{R}$, find the value of $(h^{-1})'(y)$ at the points corresponding to $x = 0, 1, -1$.

(10) [3pt] (6.1.16) Given that the restriction of the tangent function tan to $I = (-\pi/2, \pi/2)$ is strictly increasing and $\tan(I) = \mathbb{R}$, let $\arctan : \mathbb{R} \to \mathbb{R}$ be the function inverse to the restriction of tan to I. Show that arctan is differentiable on $\mathbb{R}$ and $(\arctan y)' = (1 + y^2)^{-1}$ for $y \in \mathbb{R}$.